Stability conditions on surfaces, contractions of curves, and moduli spaces

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Slope-semistability

If C is a curve, and E is a vector bundle on C, we set its *slope* as

$$\mu(E) = \frac{\deg E}{\operatorname{rk} E}.$$

We say that E is *semistable* if $\mu(E') \leq \mu(E)$ for any subsheaf $0 \neq E' \subset E$.

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Theorem (Harder–Narasimhan)

If E is a vector bundle on C, there exists a canonical filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E,$$

such that each quotient $F_i = E_i/E_{i-1}$ is semistable, and $\mu(F_1) > \mu(F_2) > \cdots > \mu(F_n)$.

Bridgeland stability

Definition

A *Bridgeland stability condition* on a variety X is $\sigma = (Z, \mathcal{A})$:

1. $\mathcal{A} \subset \mathrm{D}^b(X)$ is a heart of a bounded t-structure,
2. $Z\colon K^{\mathrm{num}}(X) \to \mathbb{C}$ is a *central charge*.

We impose:

1. Z maps $\mathcal{A}$ to the upper half-space.
2. Filtrations by semistable objects with respect to

$$\mu_\sigma(E) = \frac{-\operatorname{Re} Z(E)}{\operatorname{Im} Z(E)}.$$

Example

If C is a curve: take $\mathcal{A} = \mathrm{Coh}(C)$, $Z(E) = -\deg E + i \operatorname{rk} E$.

Stability manifold

Theorem (Bridgeland, 2007)

The set of stability conditions $\mathrm{Stab}(X)$ *carries a natural topology, and the forgetful map*

$$\mathcal{Z}\colon \mathrm{Stab}(X) \to \mathrm{Hom}_{\mathbb{Z}}(K(X), \mathbb{C}), \qquad \sigma = (Z, \mathcal{A}) \mapsto Z$$

is a local homeomorphism.

Upshot: $\mathrm{Stab}(X)$ is a complex manifold.

A subtle point

Conjecture

If X is a smooth, projective variety, then $\mathrm{Stab}(X) \neq \varnothing$.

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What do we know about $\mathrm{Stab}(X)$?

- $\dim X = 1$: Completely understood (Bridgeland, Macrì, Okada).
- $\dim X = 3$: Many known examples: Fano, Abelian, some CY3, products of curves, ...
- $\dim X \geq 4$: Only sporadic examples.

Technical issue: constructing hearts is hard!

Proposition (Toda)

If $\dim X \geq 2$, *then there is no stability condition with heart* $\mathrm{Coh}\, X$.

On surfaces

From now on: $X = S$ is a surface.

Theorem (Arcara–Bertram, 2013)

For any $\beta \in \mathrm{NS}(S)_{\mathbb{R}}$, $\omega \in \mathrm{Amp}(S)_{\mathbb{R}}$, *there is a stability condition* $\sigma_{\beta,\omega} = (Z_{\beta,\omega}, \mathcal{A}_{\beta,\omega}) \in \mathrm{Stab}(S)$, *with*

$$Z_{\beta,\omega}(-) = -\mathrm{ch}_2^{\beta}(-) + \frac{\omega^2}{2}\mathrm{ch}_0(-) + i\omega.\mathrm{ch}_1^{\beta}(-).$$

and $\mathrm{ch}^{\beta}(E) = \mathrm{ch}(E).e^{-\beta}$.

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Note: $\mathcal{A}_{\beta,\omega}$ is a *tilt* of $\mathrm{Coh}(S)$. It depends on the ray $\mathbb{R}_{>0}\omega$, and on $\beta.\omega/\sqrt{\omega^2}$.

Question

Describe the limit points of $\{\sigma_{\beta,\omega}\}_{\beta,\omega}$.

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We write $\overline{\sigma}_{\beta,\lambda}$ to denote a limit point with central charge $Z_{\beta,\lambda}$. By continuity: $\lambda \in \mathrm{Nef}(S)_{\mathbb{R}}$.

Today: Focus on the case $\lambda = f^*\eta$, $f : S \to T$ birational, T normal, projective surface.

Known cases

This question has been tackled extensively in various situations:

- (Bridgeland, 2008) S a K3 surface, and T with ADE singularities.
- (Toda, 2013–2014) $f: S \to T$ blow-up of a smooth point.
- (Tramel–Xia, 2022) $f: S \to T$ contracting a single $(-n)$-curve.
- (Chou, 2024) $f: S \to T$ contracting a single ADE chain.

Existence

Theorem (V., 2025)

Let $f\colon S \to T$ be as before. Assume that each connected component of $\mathrm{Exc}(f)$ is:

1. *a smooth, rational curve;*
2. *an A_n chain; or*
3. *a chain $C_1 \cup \cdots \cup C_n$ of smooth rational curves, $n \geq 2$ with $C_1^2, C_r^2 \leq -2$, $C_i^2 \leq -3$ otherwise.*

Then, there exists $U \subset \mathrm{NS}(S)_{\mathbb{R}}$ open such that $\overline{\sigma}_{\beta,f^*\eta}$ exists for any $\beta \in U$ and any $\eta \in \mathrm{Amp}(T)_{\mathbb{R}}$.

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Two key features:

- New singularities allowed for T.
- Many singularities simultaneously.

Other singularities?

Note from the previous theorem that T is only allowed to have certain cyclic quotient singularities.

Question

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Theorem (V., 2025)

Let $f\colon S \to T$ be as before. Assume that $\overline{\sigma}_{\beta,f^*\eta}$ exists in $\mathrm{Stab}(S)$ for some $\beta \in \mathrm{NS}(S)_{\mathbb{R}}$, $\eta \in \mathrm{Amp}(T)_{\mathbb{R}}$. Then, we have that $\mathrm{Exc}(f)$ does not contain smooth curves of positive genus.

Idea of the proof

In the previous results, the key difficulty was to *construct* the heart $\overline{\mathcal{A}}_{\beta,f^*\eta}$ for $\overline{\sigma}_{\beta,f^*\eta}$. Two issues:

- The construction is complicated! (Requires *double tilting*.)
- Not suitable to prove *non-existence*.

Idea of the proof

In the previous results, the key difficulty was to *construct* the heart $\overline{\mathcal{A}}_{\beta,f^*\eta}$ for $\overline{\sigma}_{\beta,f^*\eta}$. Two issues:

- The construction is complicated! (Requires *double tilting*.)
- Not suitable to prove *non-existence*.

Instead: We show that there is *skewed heart* $\mathcal{C}_{\beta,V}$, depending *only* on β and $V = \omega^2$, that is shared among all $\sigma_{\beta,\omega}$ with $\omega^2 = V$.

As $V = (f^*\eta)^2 > 0$, we get that $\mathcal{C}_{\beta,V}$ is also the heart for $\overline{\sigma}_{\beta,f^*\eta}$, *if it exists*.

Moduli spaces

Given $\sigma = (Z, \mathcal{A})$, we get a notion of *semistability* on $\mathcal{A}$, and so on $\mathrm{D}^b(X)$. Fix $v \in K^{num}(X)$, and consider

$$\{E : E \in \mathcal{A},\ \sigma\text{-semistable of class } v\}.$$

Extend this to families: an object $\mathscr{E} \in \mathrm{D}^b(X \times T)$ such that $\mathscr{E}|_{X \times t}$ is σ-semistable of class v for $t \in T$. This defines a stack $\mathfrak{M}_\sigma(v)$.

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Theorem (Lieblich, Toda, Alper–Halpern-Leistner–Heinloth, ...)

Under mild assumptions, $\mathfrak{M}_\sigma(v)$ is an algebraic stack admitting a proper good moduli space $M_\sigma(v)$.

Question

Describe the moduli spaces $M_\sigma(v)$.

Wall-and-chamber

Usually, there is a special $\sigma_0 \in \mathrm{Stab}(X)$ such that $M_{\sigma_0}(v)$ is well-understood: *large volume limit*, *geometric chamber*. How to relate $M_\sigma(v)$ and $M_{\sigma_0}(v)$?

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Theorem (Bridgeland, 2008)

There is a wall-and-chamber decomposition on $\mathrm{Stab}(X)$*: a collection of walls* $\{W_i\}$*, dividing* $\mathrm{Stab}(X)$ *into chambers. If* σ, σ' *lie in the same chamber, then* $M_\sigma(v) = M_{\sigma'}(v)$*.*

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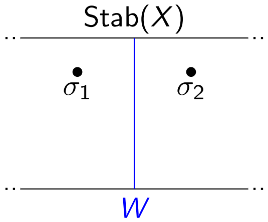
Strategy:

1. Pick a path σ_t from σ_0 to $\sigma_1 = \sigma$.
2. Analyze what happens at each wall.

At a (simple) wall

Each wall is associated to a *decomposition* $v = u + w$.



- On σ_1: $\mu_1(u) < \mu_1(w)$, only $0 \to U \to E_1 \to W \to 0$ allowed.
- On σ_2: $\mu_2(u) > \mu_2(w)$, only $0 \to W \to E_2 \to U \to 0$ allowed.

Upshot: To go from M_{σ_1} to M_{σ_2} we remove the $\{E_1\}$ and add the $\{E_2\}$.

Technical issue: How to describe this at the level of schemes:

- For each $U \in M_\sigma(u)$, $W \in M_\sigma(w)$, look at $\mathbb{P}\,\mathrm{Ext}^1(U, W)$. This can often be constructed in families.
- For each irreducible component $N \subset M_{\sigma_1}(v)$: blow-up the destabilized locus, perform an *elementary modification*.
- Glue the pieces appropriately.

Tramel–Xia

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Fact

There is a single wall passing through $\overline{\sigma}_{\beta,f^*\eta}$ with respect to $v = [pt]$.

On one side: geometric chamber

$$0 \to \mathscr{O}_C \to \mathscr{O}_p \to \mathscr{O}_C(-1)[1] \to 0.$$

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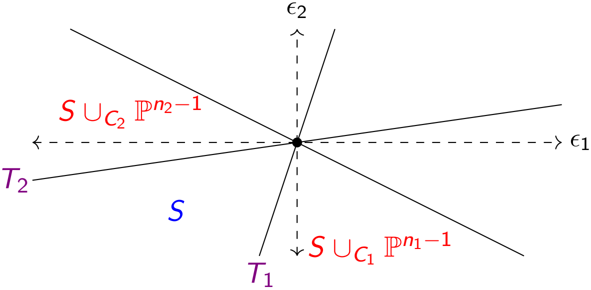
So, on the other side:

$$0 \to \mathscr{O}_C(-1)[1] \to E' \to \mathscr{O}_C \to 0.$$

These are parametrized by $\mathbb{P}\,\mathrm{Ext}^2(\mathscr{O}_C, \mathscr{O}_C(-1)) \cong \mathbb{P}^{n-1}$.

Intersecting curves

Now, assume that $f\colon S \to T$ contracts two smooth, rational curves C_1, C_2 intersecting on a point. We construct stability conditions $\sigma_{\epsilon_1,\epsilon_2}$ by deforming $\overline{\sigma}_{\beta,f^*\eta}$.



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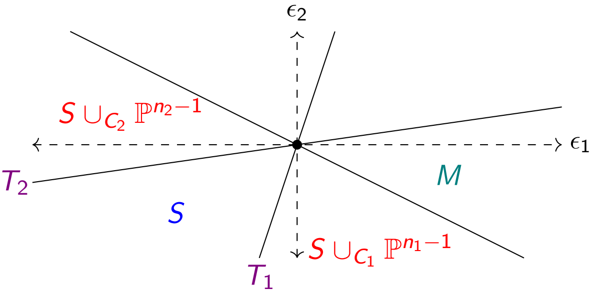

What about the other chambers?

Theorem (V., 2025)

The moduli space M has three irreducible components: S, $\mathrm{Bl}_{pt}\mathbb{P}^{n_1-1}$, and $\mathbb{P}^{n_1+n_2-3}$, glued as follows:

- *The exceptional divisor $E \subset \mathrm{Bl}_{pt}\mathbb{P}^{n_1-1}$ glues as a linear subspace of $\mathbb{P}^{n_1+n_2-3}$.*
- *The curve $C_1 \in S$ glues in $\mathrm{Bl}_{pt}\mathbb{P}^{n_1-1}$ as the strict transform of a rational normal curve passing through the blown-up point.*
- *The curve $C_2 \in S$ glues in $\mathbb{P}^{n_1+n_2-1}$ as a rational normal curve in a complementary subspace of E.*

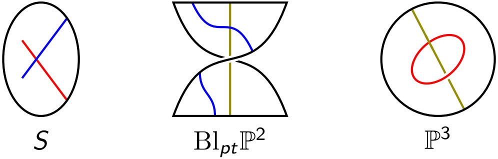

Figure: Irreducible components for $n_1 = n_2 = 3$.

A subtle point

So far: a moduli space M, together with a map $M \to M_\sigma(v)$, bijective on closed points. How to decide whether this is an isomorphism?

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If $M_\sigma(v)$ is *reduced*, this is easy: suffices to check $M \to M_\sigma(v)$ induces isomorphisms on tangent vectors. Upshot: need to understand local structure.

Deformation theory I

Start by looking at the *stack* $\mathfrak{M}_\sigma(v)$. We want to understand infinitesimal deformations of $E \in \mathrm{D}^b(X)$ σ-semistable.

Proposition (Lieblich, 2006)

Assume that E is gluable. Then:

- *Automorphisms:* $\mathrm{Aut}(E) \subset \mathrm{Hom}(E, E)$.
- *Tangent space:* $\mathrm{Ext}^1(E, E)$.
- *Obstruction space:* $\mathrm{Ext}^2(E, E)$.

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If $\mathrm{Ext}^2(E, E) = 0$, this is enough to understand the local picture. In general: deformations are controlled by a formal map

$$\kappa \colon \widehat{\mathrm{Ext}^1(E, E)} \to \widehat{\mathrm{Ext}^2(E, E)}.$$

Differential graded Lie algebras

The map κ is determined by the *differential graded Lie algebra* $R\operatorname{Hom}(E, E)$.

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How to compute it? Various options:

- Replace E with a complex of injectives $I^\bullet$; look at $\mathrm{Hom}(I^\bullet, I^\bullet)$.
- Dolbeault resolution.

In our case: Čech cocycles.

Deformation theory II

So far: local structure of the *stack* $\mathfrak{M}_\sigma(v)$. How do we go back to the *algebraic space* $M_\sigma(v)$?

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Idea (Luna, Alper–Hall–Rydh): "étale slice"

$$p \in M \quad \longleftrightarrow \quad E \in \mathrm{D}^b(X) \text{ polystable}$$

$$\text{Local structure} \quad \longleftrightarrow \quad R\,\mathrm{Hom}(E, E)$$

$$\hat{\mathscr{O}}_{M,p} \quad \longleftrightarrow \quad (\mathbb{C}[[\mathrm{Ext}^1(E,E)]]/I)^{\mathrm{Aut}(E)}$$

Proposition (V., 2025)

In the previous setup, let $E \in \mathrm{D}^b(S)$ be the object corresponding to $p = \mathbb{P}^{n_1+n_2-3} \cap S \cap \mathrm{Bl}_{pt}\mathbb{P}^{n_1-1} \subset M$. Then

$$\hat{\mathscr{O}}_{M,p} \cong \frac{\mathbb{C}[[p_1, \ldots, p_{n_1-2}, q_1, \ldots, q_{n_2-1}, r]]}{(p_1 q_1 r, \ldots, p_{n_1-2} q_1 r, q_2 r, \ldots, q_{n_2-1} r)}.$$

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In particular: M is reduced at p. Here $\hat{\mathscr{O}}_{M,p}$ is *not* cut out by quadrics!

Future directions

- Sharp conditions for singularities on T.
- What about the rest of singularities?
- Relation with stability conditions on T.

Thank you!